THE

ASSURANCE MAGAZINE.

On the Values of Annuities which are to pay certain given Rates of Interest on the Purchase Money during the whole term of their continuance, and to replace their Original Values, on their expiration, at certain other *given Rates. By* PETER HARDY, F.R.S., *one of the Vice-Presidents of the Institute of Actuaries of Great Britain and Ireland, and Actuary to the London Assurance Corporation.*

[Read before the Institute of Actuaries, 25th November, 1850.]

NOTWITHSTANDING the very large amount of leasehold property which in the course of every year is bought and sold in this country, and notwithstanding the extensive transactions—of almost hourly occurrence—in the public market, in Government and other temporary Annuities, the subject of the rate of interest which any given purchase will yield the buyer is very imperfectly understood, even by those most deeply interested in the inquiry, unless they happen to be at the same time well versed in actuarial computations.

It is not unfrequently imagined by a buyer, that if he purchase a leasehold property or a temporary Annuity at a price corresponding with the value of an Annuity at a given rate of interest (say 5 per cent.), that he has made a purchase which will pay him 5 per cent., or which, in other words, will enable him to spend 5 per cent. per annum on his outlay, and at the same time replace his capital undiminished at the expiration of the term.

This is a grave error, and very frequently leads to serious inconvenience. If a purchaser buy an Annuity for a term of years according to a 5 per cent. Table, it is absolutely essential that the surplus of the Annuity over and above the interest on the purchase-money should be annually re-invested in some fund which will also yield a clear 5 per cent., otherwise the buyer's expectation of replacing his capital at the expiration of the term will not be realized.

This is invariably so whenever the term for which the Annuity is granted exceeds a single year; in which case only, as Annuities are supposed (unless otherwise expressed) to be payable at the expiration of each year, the surplus of a single year's Annuity remaining after deducting the interest for a year, will be sufficient to replace the capital originally

expended in the purchase. For instance, suppose an Annuity of £10,000 for one year to be purchased, according to a 5 per cent. valuation,

The cost will be	£9,523	16	0
At the end of the year there will be due for interest at 5 per cent. on this sum	476	4	0
Making a total of	£10,000	0	0
Which will be exactly met by the Annuity to be then received	£10,000	0	0

But an Annuity of a similar amount purchased for two years will not exhibit the same equation.

An Annuity of £10,000 for two years, purchased according to a 5 per cent. valuation, will cost about £18,594.

At the expiration of the first year the purchaser will receive one instalment	£10,000	0	0
Out of which he will take, as one year's interest on his capital laid out, £18,594	929	14	0
Leaving surplus Annuity, to be put aside as a portion of the capital repaid	£9,070	6	0

To make this example an extreme case, we will assume that no interest is made of this sum—that no investment of it is attempted; but that it is merely placed, for the sake of custody until the completion of the transaction, in a banker's hands.

At the end of the second year the purchaser will receive the second and final instalment of his Annuity	£10,000	0	0
Out of which he again takes another year's interest on his £18,594 originally laid out . . .	929	14	0
Leaving as before a second sum of . . .	9,070	6	0
Which being added to the former balance . .	9,070	6	0
Makes a total capital of	£18,140	12	0
Which does not replace by	453	8	3
The capital originally expended	£18,594	0	0

The deficiency is manifestly the exact amount of one year's interest at 5 per cent. on the first balance of the Annuity (£9,070. 6*s.*) permitted to remain for one year unproductive in the banker's custody.

From an examination of the foregoing example, two considerations obviously suggest themselves, viz.: 1. That the instalments of repaid capital—or surplus Annuity, as they may more properly be termed—must be re-invested as they are received; and, 2. That they must be made to yield exactly the same annual rate of interest as that produced by the original investment.

The purchase, therefore, of a temporary Annuity, according to a 5 per cent. valuation, merely implies, that an interest of 5 per cent. is annually

realized on those portions of the original purchase-money which remain from time to time in the investment.

It is, however, obvious that the value of an Annuity certain for a term of years, may be so calculated as to admit of the purchaser making 5 per cent., or indeed at his option any other given rate of interest, on his outlay, during the entire term of the Annuity, and yet enable him to replace that outlay at the expiration of the term, by the accumulation of the annual surplus of the Annuity at some lower rate, say 4 or 3 per cent.

It would, moreover, seem highly desirable that a set of Tables should be calculated on this basis, to enable temporary Annuities and leasehold properties to be purchased in accordance with these views. It is nearly a quarter of a century since Mr. Griffith Davies, in his Tables for Life Contingencies, published in 1825, gave—I imagine for the first time—a Table showing the value of an Annuity on a single life which was to pay the purchaser 5, 6, or 7 per cent. on his outlay, and to replace the original capital at 3 per cent., that is to say, according to the 3 per cent. Northampton Rates. With, I believe, the single exception of Mr. Benwell,* who wrote in the year 1831 a few pages on this subject, it does not appear to have occurred to any of the actuaries who have from time to time been in the habit of employing that very useful Table, that the principle on which it was computed was equally applicable to Annuities certain until so lately as the year 1849, when it was shown by Mr. Jellicoe, one of the Vice-Presidents of the Institute of Actuaries, in a Lecture which he delivered before the members of that body, "that in the cases of temporary Annuities certain the present values as given by the Tables were not adapted to practical purposes, inasmuch as in order to reproduce the capital at the end of the term, the necessity arose of improving the portions of it returned from year to year, at the rate of interest proposed to be made in the interval."†

The idea of constructing a set of Tables in conformity with the foregoing views, was suggested to me, some two or three years ago, by Mr. William Drummond, solicitor, of Croydon. My want of leisure for the inquiry did not, however, admit of my immediately working out Mr. Jellicoe's hint, or following up Mr. Drummond's suggestions; and I have from time to time deferred my proposed investigation of the subject. With the assistance, however, of my friend Mr. Edgar Sharpe, A. I. A., of the London Assurance Corporation, I have prepared a small set of Tables, which will be found, I think, useful, as embracing those rates and terms most likely to present themselves in practice; and I now submit these Tables, together with the accompanying remarks and investigations, to the Institute of Actuaries. The following is an algebraical investigation of the question:—

* It is, however, due to Mr. Benwell to state, that his work above referred to distinctly treats the subjects now under consideration, but in a style so little happy, and so involved, that the merit of the actuary is apt to be overlooked in the obscurity of the writer. Mr. Benwell, moreover, computed some Tables of a character similar to those appended to this paper: two of the columns in Table VII. of his little work are identical with Value Columns 7 and 11 in the accompanying Table; which Table was, however, independently calculated,—indeed, I never saw Mr. Benwell's book until the present paper had been written for some weeks, and was actually in type.

† See an account of this Lecture in the *Post Magazine* for 10th March, 1849.

Problem,

To determine the present value of an Annuity certain of £1 per annum for n years, which is to pay, during its continuance, a given rate of interest on the original purchase-money, and to replace that purchase-money at the expiration of the term at a different rate of interest.

Solution.

The annual payments of the Annuity being each = £1, let i' = the rate of interest which the purchaser intends to make on each £1 of his investment, or, as it may be termed, the *remunerative rate.* Let $(r-1)=$ the rate of interest at which the purchaser expects to accumulate the surplus Annuity, in order to replace the original capital, or, as it may be termed, *the accumulative rate.* Let $\dfrac{r^n-1}{r-1}$ = the amount of an Annuity of £1 per ann. for n years forborne and accumulated at $(r-1)$ rate of interest (*see* Author's 'Doctrine of Interest, Simple and Compound,' 1839. Prob. II., Sec. II.), and put V = the required value.

Now it is obvious that

Vi' = the purchaser's annual interest;

$1-Vi'$ = the surplus Annuity to be accumulated, so that in n years it may reproduce V.

If £1 per ann. in n years will accumulate into $\dfrac{r^n-1}{r-1}$, then

$V=(1-Vi')\dfrac{r^n-1}{r-1}$; and if, for the sake of simplicity, we represent $\dfrac{r^n-1}{r-1}$ by a single symbol, say $\mathfrak{A}$, we shall have

$$V=(1-Vi')\,\mathfrak{A}$$
$$V=\mathfrak{A}-Vi'\mathfrak{A}$$
$$V+Vi'\mathfrak{A}=\mathfrak{A}$$
$$V(1+i'\mathfrak{A})=\mathfrak{A}$$
$$V=\frac{\mathfrak{A}}{1+i'\mathfrak{A}}.$$

The correctness of this solution may be readily ascertained, because it is manifest from what has been previously said, viz. that in order to realize the rate of interest at which the valuation was made, it is necessary that the surplus Annuity should be invested in, or accumulate at, the same rate of interest, so that if the accumulative rate in the above equations be made equal to the remunerative rate, that is, if $i'=(r-1)$, then V, the final value produced, should be equal to the ordinary value of an Annuity for n years at $(r-1)$ rate of interest, that is to say,

$$V=\frac{\mathfrak{A}}{1+i'\mathfrak{A}}=\frac{\dfrac{r^n-1}{r-1}}{1+(r-1)\times\dfrac{r^n-1}{r-1}}\ \therefore\ V=\frac{r^n-1}{r-1}\times\frac{1}{r^n},$$

which is the expression for the value required. (*See* Author's 'Doctrine of Interest.' Prob. V., Sec. II.)

The equation $V=(1-Vi')\mathfrak{A}$, naturally suggests a very simple method of solving the problem previously referred to, on which Mr. Griffith Davies has founded his Table showing the value of an Annuity on a Single Life, allowing a given rate of interest and the premium for assurance.

It is to be borne in mind that every annual premium (p) payable during the existence of a given life, for an assurance of £1 on death, is an annual sum which (on an average of cases) will provide, or in other words accumulate into, the sum assured on the extinction of the life in question.

Now Mr. Davies' Table is so constructed, that the value given for the Annuity is to be reproduced by the assurance on the extinction of the life; and as the purchaser is to have his interest on this value out of every £1 received as Annuity, it is obvious that the premium which reproduces the value or the assurance, is the difference between the said interest and £1, in like manner as the difference between £1 and Vi, in the above equation, reproduces V by accumulation; but there is a difference in the conditions under which the Annuity certain is reproduced at the expiration of the term, and those under which the sum assured is reproduced on the extinction of the life. V in the first case not only represents the original outlay, but also represents the sum to be ultimately reproduced; but in the case of a Life Annuity, V does not represent the original outlay, inasmuch as the premium (p) being made payable at the commencement of each year, the outlay, V, must be increased by the first premium expended: neither does V, increased by the amount of this premium, represent the sum which is to be insured or reproduced, because one year's interest on both V and the first payment of the premium must be secured by the assurance, inasmuch as from the nature of a Life Annuity, which is made payable at the end only of each year, to the termination of which the annuitant survives, no Annuity is received for the year in which the life drops, consequently in that year the purchaser would lose both premium and interest, were it not secured to him by the assurance. We have, however, seen that £1 = both premium and interest, and therefore $V+1=$ the amount to be assured; and since $V+1=$ the sum to be assured, $p\ (V+1)=$ the premium actually expended, as an original outlay in the first year, and as the surplus of the Annuity in all subsequent years, therefore the original outlay $=V+p(V+1)$, and the purchaser's interest thereon $=Vi'+pi'(V+1)$; and since as $p : 1 :: 1 : \frac{1}{p}$, we shall have instead of

$$V=(1-Vi')\mathfrak{A},$$

the corresponding equation,

$$V+1=(1-Vi'-pi'\overline{V+1})\frac{1}{p}$$

$$V\left(\frac{p+pi'+i'}{p}\right)=\frac{1-p-pi'}{p}$$

$$V=\frac{1-p(1+i')}{p(1+i')+i'},$$

which will be found to correspond with the solution given by Mr. David Jones in his work on Annuities, &c., vol. i. p. 190, and to be equivalent to the more simple and far more elegant expression given by Mr. Davies for the solution of the same problem in his unpublished work, chap. iv. pp. 250, 251, viz.:—

$$\mathrm{V}=\frac{1}{d+p}-1, \text{ where } d=\frac{r-1}{r}.$$

It is apparent, on a little consideration, that the foregoing problem is in some measure complicated by the circumstance that the value of the Annuity and the sum to be assured both differ from the sum actually expended by the purchaser.

For instance, the value of the Annuity is V,
The sum to be assured is $\mathrm{V}+1$,
The sum expended is $\mathrm{V}+p\,.\,(\mathrm{V}+1)$.

The necessity for these differences, as I have already pointed out, arises from premiums being made payable in advance at the beginning of each year, and also from the circumstance of Annuities being *not payable* for the year in which the Annuitant dies.

If an Annuity, however, in addition to being payable at the end of each year, if the Annuitant be alive, were also to be made payable at the end of the year in which he died—that is, if the Annuity were to be made *payable for the year of death;* and further, if the annual premium were to be made payable at the end instead of at the commencement of each year and *also for the year of death,* in such case it is evident that V will equally represent the value of the Annuity, the sum actually expended, and the amount to be assured, and the problem would then become quite as simple in its form as that of an Annuity certain. For example, the value of an ordinary Annuity on a single life aged A years will be represented in my *Notation* (1840) by $\mathrm{I}\mathrm{A}$, and the value of a reversion of £1 payable on the extinction of the same life by the symbol $\mathrm{A}\mathrm{I}\mathring{\mathrm{i}}$. Now it is obvious that the value of an Annuity which is made payable for the year of death is $\mathrm{I}\mathrm{A}+\mathrm{A}\mathrm{I}\mathring{\mathrm{i}}$. That is to say, the value of the ordinary Annuity is increased by the value of £1 payable at the end of the year in which A dies, and since $\mathrm{I}\mathrm{A}+\mathrm{A}\mathrm{I}\mathring{\mathrm{i}}=\dfrac{\mathrm{I}\mathrm{A}+1}{r}$,

the annual premium (ϕ) payable *at the end instead of the beginning of the year*, and for the year of death, will be $\dfrac{\mathrm{A}\mathrm{I}\mathring{\mathrm{i}}}{\mathrm{I}\mathrm{A}+\mathrm{A}\mathrm{I}\mathring{\mathrm{i}}}=\phi$;

or, what is the same thing, $\dfrac{\mathrm{A}\mathrm{I}\mathring{\mathrm{i}}}{\dfrac{\mathrm{I}\mathrm{A}+1}{r}}=\phi$,

and $\dfrac{A\mathrm{I}\mathring{\mathrm{i}}}{\mathrm{I}_{A+1}} = \dfrac{\phi}{r}$:

therefore, as $\dfrac{A\mathrm{I}\mathring{\mathrm{i}}}{\mathrm{I}_{A+1}} = p,\ p = \dfrac{\phi}{r}$, and $\phi = pr$,

and instead of $\quad V = (1 - Vi')\,\mathfrak{A}$,

we shall have $\quad V = (1 - Vi')\dfrac{1}{pr}$

$$V = \frac{1}{pr + i'}.$$

To resume, however, the subject more immediately before us in this paper, The equation

$$V = \frac{\mathfrak{A}}{1 + i'\mathfrak{A}}$$

offers a very simple rule for the construction of a set of Tables. The rule itself may be thus given in words at length:—

"Multiply the amount of an Annuity forborne for n years at the accumulative rate into the remunerative rate, add unity to the product, and multiply the reciprocal of the sum into the amount of the Annuity forborne, as above, and the product will give the value of the Annuity required."

EXAMPLE.

Required the present value of an Annuity of £1 per annum for 20 years, the purchaser to make 5 per cent. per ann. interest of his outlay, and to replace his capital by the investment of his surplus Annuity at 3 per cent.

$$\begin{aligned} \text{Here the Annuity} &= 1 \\ 'i &= \cdot 05 \\ (r-1) &= \cdot 03 \end{aligned}$$

and by Tab. III., Author's 'Doctrine of Int.,' $\mathfrak{A}$ at 3 per cent. = 26·8703.

$$\begin{array}{rcl} \lambda\ 26\cdot8703 & = & 1\cdot4292677 \\ \cdot05 & & \\ \hline 1\cdot343515 & & \\ 1 & & \\ \hline \lambda\ 2\cdot343515 & = & \underline{0\cdot3698650} \\ & & 1\cdot0594027 = \lambda\ 11\cdot466 = \text{value.} \end{array}$$

TABLE *showing the Present Value of an Annuity of £1 per annum, for a given number of Years certain, supposing the Purchaser thereof to take out of the Annuity* £5 *per cent.,* £6 *per cent., or* £7 *per cent., per annum, as an available Interest on his Purchase-money or Capital advanced, while he is only enabled to re-invest the* SURPLUS *of the Annuity beyond the available Interest, so as to make* 3 *per cent.,* 3½ *per cent.,* 4 *per cent., or* 5 *per cent. thereof.*

YEARS.	Interest to be 5 per cent.			Interest to be 6 per cent.				Interest to be 7 per cent.			
	The Re-investments to be made at the Rate of			The Re-investments to be made at the Rate of				The Re-investments to be made at the Rate of			
	3 per cent.	3½ per cent.	4 per cent.	3 per cent.	3½ per cent.	4 per cent.	5 per cent.	3 per cent.	3½ per cent.	4 per cent.	5 per cent.
	Value.	Value.	Value.	Value.	Value.	Value.	Value.	Value.	Value.	Value.	Value.
1	·952	·952	·952	·943	·943	·943	·943	·934	·934	·934	·934
2	1·842	1·847	1·851	1·809	1·813	1·817	1·825	1·777	1·780	1·785	1·792
3	2·677	2·688	2·700	2·607	2·618	2·629	2·651	2·541	2·551	2·561	2·582
4	3·460	3·481	3·502	3·344	3·364	3·384	3·424	3·236	3·254	3·273	3·311
5	4·195	4·228	4·262	4·026	4·057	4·088	4·149	3·870	3·899	3·927	3·984
6	4·887	4·934	4·981	4·659	4·702	4·744	4·830	4·452	4·491	4·529	4·608
7	5·540	5·601	5·662	5·238	5·304	5·359	5·470	4·987	5·036	5·086	5·186
8	6·155	6·231	6·308	5·724	5·866	5·933	6·071	5·481	5·540	5·601	5·723
9	6·737	6·828	6·920	6·311	6·391	6·472	6·636	5·937	5·871	6·079	6·223
10	7·287	7·394	7·502	6·792	6·885	6·979	7·168	6·359	6·411	6·523	6·688
11	7·807	7·930	8·054	7·242	7·347	7·454	7·669	6·753	6·844	6·937	7·122
12	8·301	8·439	8·579	7·665	7·782	7·901	8·141	7·119	7·220	7·322	7·529
13	8·769	8·923	9·078	8·062	8·192	8·323	8·586	7·400	7·572	7.683	7·907
14	9·214	9·383	9·554	8·437	8·578	8·720	9·007	7·780	7·900	8·021	8·262
15	9·636	9·820	10·006	8·789	8·942	9·095	9·403	8·079	8·208	8·337	8·595
16	10·038	10·237	10·436	9·122	9·286	9·450	9·778	8·360	8·497	8·633	8·907
17	10·423	10·634	10·847	9·438	9·611	9·784	10·132	8·624	8·768	8·912	9·199
18	10·787	11·011	11·237	9·736	9·919	10·102	10·467	8·872	9·003	9·175	9·474
19	11·134	11·372	11·610	10·019	10·210	10·402	10·782	9·106	9·264	9·421	9·732
20	11·466	11·715	11·965	10·286	10·487	10·686	11·081	9·326	9·491	9·654	9·975
21	11·783	12·043	12·303	10·541	10·749	10·955	11·364	9·535	9·705	9·873	10·206
22	12·085	12·356	12·626	10·782	10·997	11·211	11·632	9·732	9·910	10·084	10·420
23	12·375	12·655	12·936	11·012	11·234	11·454	11·885	9·919	10·099	10·277	10·623
24	12·651	12·941	13·230	11·231	11·459	11·684	12·125	10·096	10·280	10·464	10·814
25	12·916	13·215	13·512	11·438	11·672	11·903	12·353	10·264	10·452	10·637	10·995
26	13·169	13·476	13·781	11·637	11·876	12·111	12·569	10·424	10·615	10·803	11·165
27	13·412	13·727	14·037	11·825	12·070	12·310	12·773	10·575	10·770	10·960	11·326
28	13·644	13·966	14·283	12·006	12·254	12·527	12·967	10·719	10·917	11·110	11·478
29	13·867	14·195	14·518	12·178	12·431	12·678	13·150	10·856	11·056	11·251	11·622
30	14·081	14·415	14·743	12·343	12·599	12·849	13·324	10·987	11·189	11·386	11·757
31	14·286	14·626	14·958	12·500	12·760	13·012	13·490	11·111	11·316	11·514	11·886
32	14·483	14·828	15·164	12·651	12·913	13·167	13·646	11·231	11·437	11·635	12·007
33	14·673	15·021	15·361	12·795	13·060	13·315	13·795	11·344	11·551	11·751	12·123
34	14·854	15·207	15·549	12·933	13·200	13·456	13·936	11·452	11·660	11·860	12·232
35	15·029	15·385	15·729	13·065	13·334	13·591	14·070	11·556	11·765	11·965	12·335
36	15·197	15·556	15·902	13·192	13·462	13·720	14·198	11·654	11·865	12·065	12·432
37	15·359	15·720	16·067	13·314	13·584	13·843	14·319	11·750	11·959	12·160	12·525
38	15·514	15·878	16·225	13·430	13·702	13·961	14·433	11·840	12·050	12·250	12·613
39	15·663	16·028	16·377	13·542	13·815	14·073	14·542	11·927	12·138	12·337	12·696
40	15·808	16·174	16·523	13·650	13·922	14·180	14·649	12·011	12·221	12·419	12·778
41	15·946	16·314	16·662	13·753	14·025	14·282	14·744	12·090	12·300	12·497	12·849
42	16·080	16·448	16·795	13·853	14·125	14·380	14·838	12·167	12·377	12·572	12·921
43	16·208	16·577	16·925	13·947	14·220	14·474	14·927	12·240	12·450	12·644	12·989
44	16·332	16·699	17·046	14·039	14·310	14·564	15·012	12·311	12·520	12·712	13·052
45	16·451	16·820	17·164	14·127	14·398	14·650	15·092	12·380	12·587	12·778	13·113
46	16·567	16·934	17·277	14·212	14·482	14·731	15·168	12·444	12·621	12·840	13·171
47	16·678	17·045	17·385	14·294	14·562	14·810	15·241	12·507	12·712	12·900	13·256
48	16·785	17·150	17·489	14·372	14·369	14·885	15·311	12·567	12·770	12·957	13·278
49	16·888	17·253	17·588	14·448	14·714	14·958	15·376	12·625	12·800	13·012	13·327
50	16·989	17·311	17·684	14·521	14·751	15·025	15·439	12·680	12·882	13·064	13·373
60	17·815	18·153	18·451	15·121	15·363	15·612	15·916	13·136	13·318	13·477	13·731
70	18·405	18·706	18·960	15·530	15·757	15·938	16·208	13·453	13·613	13·747	13·948
80	18·828	19·089	19·300	15·846	16·030	16·178	16·385	13·678	13·815	13·925	14·079
90	19·137	19·358	19·528	16·063	16·219	16·341	16·494	13·840	13·955	14·044	14·159
99	19·343	19·531	19·669	16·208	16·340	16·437	16·556	13·948	14·044	14·116	14·204
100	19·363	19·547	19·682	16·222	16·351	16·446	16·561	13·957	14·053	14·123	14·209

London Assurance Corporation, 7, *Royal Exchange*,
23*d September*, 1850.

PETER HARDY.